Math 105 Chapter 6: Techniques of Integration II

We will now spend some time on more advanced techniques of integration:

Trigonemetric Integrals:

The following are identities that you must know:

① $\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$ (divide by $\cos^2\theta$)

$1 + \cot^2\theta = \csc^2\theta$ (divide by $\sin^2\theta$)

② $\sin(2\theta) = 2\sin\theta\cos\theta$

③ $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$= 2\cos^2\theta - 1$

$= 1 - 2\sin^2\theta$

(② and ③: double angle formulas)

④ $\frac{d}{d\theta}\sin\theta = \cos\theta$ $\qquad$ $\frac{d}{d\theta}\cos\theta = -\sin\theta$

$\frac{d}{d\theta}\tan\theta = \sec^2\theta$ $\qquad$ $\frac{d}{d\theta}\cot\theta = -\csc^2\theta$

$\frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta$ $\qquad$ $\frac{d}{d\theta}\csc\theta = -\csc\theta\cot\theta$

⑤ $\int \tan\theta\, d\theta = \log|\sec\theta| + C$, $\int \sec\theta\, d\theta = \log|\sec\theta + \tan\theta| + C$

Now lets do some integrals.

eg $\int \sin^2\theta \, d\theta$

By (3) we have $\cos 2\theta = 1 - 2\sin^2\theta$

$$\Rightarrow \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

Thus
$$\int \sin^2\theta \, d\theta = \int \frac{1}{2} - \frac{1}{2}\cos 2\theta \, d\theta$$
$$= \frac{1}{2}\int d\theta - \frac{1}{2}\int \cos 2\theta \, d\theta$$
$$= \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$$

excercise: Find $\int \cos^2\theta \, d\theta$

eg $\int \sin^2(2\theta)\sin^3\theta \, d\theta$

$$= \int (2\sin\theta\cos\theta)^2 \sin^3\theta \, d\theta \quad \text{, by (2)}$$
$$= \int 4\sin^5\theta\cos^2\theta \, d\theta$$
$$= \int 4(\sin^2\theta)^2\cos^2\theta \cdot \sin\theta \, d\theta$$
$$= \int 4(1-\cos^2\theta)^2\cos^2\theta \cdot \sin\theta \, d\theta, \quad u = \cos\theta, \ \frac{du}{d\theta} = -\sin\theta \Rightarrow \sin\theta \, d\theta = -du$$
$$= -4\int (1-u^2)^2 u^2 \, du$$
$$= -4\int (1 - 2u^2 + u^4) u^2 \, du$$
$$= -4\int u^2 - 2u^4 + u^6 \, du$$

$$\Rightarrow \int \sin^2(2\theta)\sin^3\theta \, d\theta$$

$$= -4\left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right) + C$$

$$= -\frac{4}{3}u^3 + \frac{8}{5}u^5 - \frac{4}{7}u^7 + C$$

$$= -\frac{4}{3}\cos^3\theta + \frac{8}{5}\cos^5\theta - \frac{4}{7}\cos^7\theta + C$$

General strategy for dealing with $\int \sin^n x \cos^n x \, dx$:

- If n is odd, then keep one $\sin x$ isolated and turn the remaining $n-1$ sines into cosines by using (1).

$$\text{eg} \quad \int \sin^9 x \cos^4 x \, dx$$

$$= \int \sin^8 x \cos^4 x \sin x \, dx$$

$$= \int (1-\cos^2 x)^4 \cos^4 x \sin x \, dx$$

Then u-substitute $\cos x$.

- If m is odd, then keep one cosine isolated and turn the remaining $m-1$ cosines into sines by using (1).

$$\text{eg} \quad \int \sin^{21} x \cos^3 x \, dx$$

$$= \int \sin^{21} x \cos^2 x \cdot \cos x \, dx$$

$$= \int \sin^{21} x (1-\sin^2 x) \cos x \, dx$$

Then u-substitute $\sin x$.

- If both m, and n are odd then apply double angle formula ie (3) and simplify until you get the othe cases. This is not a fun case.

eg $\int \sin^2 x \cos^2 x \, dx$

$= \int \frac{1}{2}(1-\cos(2x))\frac{1}{2}(1+\cos(2x))\,dx$, by (3)

$= \frac{1}{4}\int (1-\cos 2x)(1+\cos 2x)\,dx$

$= \frac{1}{4}\int 1-\cos^2 2x \, dx$

$= \frac{1}{4}\int 1-\frac{1}{2}(1+\cos 4x)\,dx$, by (3)

$= \frac{1}{4}\int \frac{1}{2}-\frac{1}{2}\cos 4x \, dx$

$= \frac{1}{8}\int 1 - \cos 4x \, dx$

$= \frac{x}{8} - \frac{\sin 4x}{32} + C$

ex find $\int \cos^4 x \sin^2 x \, dx$

$\int \sin x \cos x \cos 2x \, dx$

eg $\int \tan^5 x \sec^2 x \, dx$

$= \int (\tan^2 x)^2 \sec x \cdot \tan x \sec x \, dx$

$= \int (\sec^2 x - 1)^2 \sec x \cdot \tan x \sec x \, dx$, $u = \sec x$, $du = \tan x \sec x \, dx$

$= \int (u^2 - 1)^2 u \, du$

$= \int u^5 - 2u^3 + u \, du$

$= \dfrac{u^6}{6} - \dfrac{u^4}{2} + \dfrac{u^2}{2} + C$

$= \dfrac{\sec^6 x}{6} - \dfrac{\sec^4 x}{2} + \dfrac{\sec^2 x}{2} + C$

eg $\int \tan^{10} x \sec^4 x \, dx$

$= \int \tan^{10} x \sec^2 x \sec^2 x \, dx$

$= \int \tan^{10} x (1 + \tan^2 x) \sec^2 x \, dx$, $u = \tan x$, $du = \sec^2 x \, dx$

$= \int u^{10} (1 + u^2) \, du$

$= \int u^{10} + u^{12} \, du$

$= \dfrac{u^{11}}{11} + \dfrac{u^{13}}{13} + C$

$= \dfrac{\tan^{11} x}{11} + \dfrac{\tan^{13} x}{13} + C$

General strategy of dealing with $\int \tan^m x \sec^n x \, dx$

- If m is odd split off $\tan x \sec x$, and turn the remaining tan's into secants using ①.

$$\text{eg } \int \tan^7 x \sec^3 x \, dx = \int (\tan^2 x)^3 \sec^2 x \tan x \sec x \, dx$$
$$= \int (\sec^2 x - 1)^3 \sec^2 x \tan x \sec x \, dx$$

Then let $u = \sec x$

- If n is even, split off $\sec^2 x$ and turn the remaining secants into tans using ①

$$\text{eg } \int \tan^3 x \sec^6 x \, dx = \int \tan^3 x (\sec^2 x)^2 \sec^2 x \, dx$$
$$= \int \tan^3 x (1 + \tan^2 x)^2 \sec^2 x \, dx$$

Then let $u = \tan x$

- If m is even and n is odd, then you pray!

You rewrite $\tan^m x$ as a bunch of secants and integrate by parts

This case sucks. A lot.

exercise: Find $\int \tan^5 x \, dx$

$$\int \tan^4 \sec^6 x \, dx$$

$$\int \frac{\sin^5 x}{\cos^8 x} \, dx$$

Partial Fractions

In this section our goal is to integrate rational functions. I.e functions of the form:

$$f(x) = \frac{p(x)}{q(x)}$$

where p, q are polynomials.

eg. $\frac{x+2}{x^2-1}$, $\frac{x^2+3x+6}{x^4+2x^2+1}$, $\frac{x^3+2x+1}{x^2+2}$

There are 2 cases we need to worry about:

1. $\deg p < \deg q$
2. $\deg p \geq \deg q$

Let's deal with the first case.

Case 1 $\deg p < \deg q$.

eg $\int \frac{1}{x^2-4}\,dx$

It is always a good idea to factor the denominator.

$$\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)}$$

Now we know how to integrate $\frac{1}{x-2}$, $\frac{1}{x+2}$, but not the product. Integration by parts is not very helpful here since

$$u = \frac{1}{x-2}, \quad dv = \frac{1}{x+2}$$

$$du = \frac{-1}{(x-2)^2} \quad v = \log|x+2|$$

$$\Rightarrow \int \frac{1}{(x-2)(x+2)}\,dx = \frac{\log|x+2|}{x-2} + \underbrace{\int \frac{\log|x+2|}{(x-2)^2}\,dx}_{\star}$$

$\star$ is clearly much worse that what we started!

So what can we do? Magic! I claim that:

$$\frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} \quad \text{for some } A, B.$$

We need to find these A, B; by finding a common denominator and equating numerators.

$$\frac{A}{x-2} + \frac{B}{x+2} = \frac{A(x+2) + B(x-2)}{(x-2)(x+2)} = \frac{1}{(x-2)(x+2)}$$

$$\Rightarrow 1 = A(x+2) + B(x-2)$$
$$= (A+B)x + (2A-2B)$$

By equating terms: $\begin{cases} A+B=0 & (1) \\ 2A-2B=1 & (2) \end{cases}$

(1) $\Rightarrow$ $A = -B$ so by (2) we have

$$-2B - 2B = 1$$

$$\Rightarrow B = \frac{-1}{4}, \quad A = \frac{1}{4}$$

So
$$\int \frac{1}{(x-2)(x+2)}\,dx = \frac{1}{4}\int \frac{1}{x-2}\,dx - \frac{1}{4}\int \frac{1}{x+2}\,dx$$
$$= \frac{1}{4}\log|x-2| - \frac{1}{4}\log|x+2| + C$$
$$= \frac{1}{4}\log\left|\frac{x-2}{x+2}\right| + C$$

The procedure of splitting our rational function into a sum of rationals is called <u>partial fractions</u>

Procedure for guessing the correct partial fractions:

Given $f(x) = \frac{p(x)}{q(x)}$, $\deg(p) < \deg(q)$

Let us first examine the case where $q(x)$ splits into linear terms. ie $q(x) = (x-a_1)^{n_1}(x-a_2)^{n_2}\cdots(x-a_n)^{n_r}$

eg $q(x) = (x-2)(x+3)^2$, $a_1 = 2$, $n_1 = 1$
$a_2 = -3$, $n_2 = 2$

(1) If $n_i = 1$ for all $i = 1, 2, \dots, r$

Then $$\frac{p(x)}{q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_r}{x-a_r}$$

eg $$\frac{3x-1}{(x-2)(x+1)(x-1)} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{x-1}$$

(2) If there is a repeated root, then you add successive powers:

eg $q(x) = (x-2)^3(x-1)$:

$$\frac{x^3+3x}{(x-2)^3(x-1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{D}{x-1}$$

eg $$\int \frac{x^2+1}{(x-1)(x+1)^2}\,dx$$

$$\frac{x^2+1}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$= \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2}$$

$$\Rightarrow x^2+1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$$= Ax^2 + 2Ax + A + Bx^2 - Bx + Bx - B + Cx - C$$

$$= (A+B)x^2 + (2A+C)x + (A-B-C)$$

$$\Rightarrow \begin{cases} A+B = 1 & (1) \\ 2A+C = 0 & (2) \\ A-B-C = 1 & (3) \end{cases}$$

$(2) \Rightarrow C = -2A$, so (3) tells us

$$A - B - (-2A) = 1 \quad \text{or} \quad 3A - B = 1 \quad (4)$$

$(1) + (4) \Rightarrow 4A = 2$

$$\Rightarrow \quad A = \frac{1}{2}$$

So $B = 1 - A = \frac{1}{2}$, $C = -2A = -1$

So $$\int \frac{x^2+1}{(x-1)(x+1)^2}\,dx = \frac{1}{2}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx - \int \frac{1}{(x+1)^2}\,dx$$

$$= \frac{1}{2}\log|x-1| + \frac{1}{2}\log|x+1| + \frac{1}{x+1} + C$$

What if doesn't factor as a bunch of linear terms? Well we pretty much do the same thing.

eg $$\int \frac{x+1}{x(x^2+1)}\,dx$$

$$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$ ← we need a degree 1 polynomial the denominator is degree 2.

Let us compute the integral first then find A, B, C.

$$\int \frac{x+1}{x(x^2+1)}\,dx = A\underbrace{\int \frac{1}{x}\,dx}_{I_1} + B\underbrace{\int \frac{x}{x^2+1}}_{I_2} + C\underbrace{\int \frac{1}{x^2+1}\,dx}_{I_3}$$

$I_1 = \log|x| + C_1$

$I_2 = \int \frac{x}{x^2+1}\,dx, \quad u = x^2+1,\ du = 2x\,dx$

$= \frac{1}{2}\int \frac{1}{u}\,du$

$= \frac{1}{2}\log|u| + C_2$

$= \frac{1}{2}\log|x^2+1| + C_2$

$I_3 = \arctan x + C_3$

$\Rightarrow \int \frac{x+1}{x(x^2+1)}\,dx = A\log|x| + \frac{B}{2}\log|x^2+1| + C\arctan x + \text{Constant}$

Now lets find A, B, C.

$$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)}$$

$\Rightarrow \quad x+1 = Ax^2 + A + Bx^2 + Cx$

$\qquad\qquad = (A+B)x^2 + Cx + A$

$$\Rightarrow \begin{cases} A+B=0 \\ C=1 \\ A=1 \end{cases}$$

Thus $A=1,\ C=1,\ B=-A=-1$

$$\Rightarrow \int \frac{x+1}{x(x^2+1)}\,dx = \log|x| - \frac{1}{2}\log|x^2+1| + \arctan x + C$$

In general if you cannot factor into linear terms it is extremely useful to complete the square. In the previous example x^2+1 was already in that form. Let do an example where we cant do that.

eg $$\int \frac{x+2}{x^2+2x+3}\,dx$$

$$x^2+2x+3 = x^2+2x+1-1+3$$
$$= (x+1)^2+2$$

$$\Rightarrow \int \frac{x+2}{x^2+2x+3}\,dx = \int \frac{x+2}{(x+1)^2+2}\,dx \quad , \ u = x+1 \ , \ du = dx$$
$$x = u-1$$

$$= \int \frac{u-1+2}{u^2+2}\,du$$

$$= \int \frac{u+1}{u^2+2}\,du$$

$$= \underbrace{\int \frac{u}{u^2+2}\,du}_{I_1} + \underbrace{\int \frac{1}{u^2+2}\,du}_{I_2}$$

$$I_1 = \int \frac{u}{u^2+2}\,du \quad , \quad w = u^2+2 \ , \ dw = 2u\,du$$

$$= \frac{1}{2}\int \frac{1}{w}\,dw$$

$$= \frac{1}{2}\log|w| + C_1$$

$$= \frac{1}{2}\log|u^2+2| + C_1$$

$$= \frac{1}{2}\log|(x+1)^2+2| + C_1$$

$$= \frac{1}{2}\log|x^2+2x+3| + C_1$$

$$I_2 = \int \frac{1}{u^2+2}\,du \quad ,$$

$$= \frac{1}{\sqrt{2}}\int \frac{\sqrt{2}}{u^2+2}\,du$$

$$= \frac{1}{\sqrt{2}}\arctan\left(\frac{u}{\sqrt{2}}\right) + C_2 \ , \quad \text{since } \frac{d}{dx}\arctan\left(\frac{x}{a}\right) = \frac{a}{x^2+a^2}$$

In our case $a = \sqrt{2}$

$$= \frac{1}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + C_2$$

$$\Rightarrow \int \frac{x+2}{x^2+2x+3}\,dx = \frac{1}{2}\log|x^2+2x+3| + \frac{1}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + C_2$$

Case 2 $\deg p \geq \deg q$

To deal with this case we long divide and then use case 1.

eg $\displaystyle\int \frac{x^3+2x^2-1}{x^2-4}\,dx$

We long divide

$$\begin{array}{r}
x+2 \quad \leftarrow \text{quotient} \\
x^2-4 \,\overline{\big)\, x^3+2x^2+0x-1} \\
\underline{x^3-0x^2-4x} \\
2x^2+4x-1 \\
\underline{2x^2+0x-8} \\
4x+7 \quad \leftarrow \text{remainder}
\end{array}$$

$$\Rightarrow \frac{x^3+2x^2-1}{x^2-4} = x+2 + \frac{4x+7}{x^2-4}$$

$$\Rightarrow \int \frac{x^3+2x^2-1}{x^2-4}\,dx = \int x+2 + \frac{4x+7}{x^2-4}\,dx$$

$$= \frac{x^2}{2} + 2x + \int \frac{4x+7}{x^2-4}\,dx$$

$$\underset{\overset{\shortparallel}{(x-2)(x+2)}}{\frac{4x+7}{x^2-4}} = \frac{A}{x-2} + \frac{B}{x+2} = \frac{A(x+2)+B(x-2)}{(x-2)(x+2)}$$

$$\Rightarrow 4x+7 = A(x+2)+B(x-2)$$
$$= (A+B)x + (2A-2B)$$

$$\Rightarrow \begin{cases} A+B = 4 & (1) \\ 2A-2B = 7 & (2) \end{cases}$$

Multiplying (1) by 2, we get

$$2A + 2B = 8 \quad (3)$$

$$(1) + (3) \Rightarrow \quad 4A = 15 \quad \text{ie} \quad A = \frac{15}{4}$$

$$(1) \Rightarrow \quad B = 4 - A = 4 - \frac{15}{4} = \frac{1}{4}$$

$$\Rightarrow \int \frac{4x+7}{x^2-4}\,dx = A\int \frac{1}{x-2}\,dx + B\int \frac{1}{x+2}\,dx$$

$$= \frac{15}{4}\log|x-2| + \frac{1}{4}\log|x+2| + C$$

Thus $\displaystyle\int \frac{x^3+2x^2-1}{x^2-4}\,dx = \frac{x^2}{2} + 2x + \frac{15}{4}\log|x-2| + \frac{1}{4}\log|x+2| + C$

Trigonometric Substitution

Recall on monday we saw how to compute various trigonometric integrals. Today we will see how those integrals open up the possibility to do even more integrals!

We will be abusing the following identities:

$$(1) \quad \sin^2 x + \cos^2 x = 1$$
$$(2) \quad \tan^2 x + 1 = \sec^2 x$$
$$(3) \quad 1 + \cot^2 x = \csc^2 x$$

With u-substitution you let u be a function of x. With trig sub we let x be a function of some trigonemtric function. Let's demonstrate with an example.

eg $$\int \frac{\sqrt{9-x^2}}{x^2}\,dx$$

We know that $\sin^2 x + \cos^2 x = 1$, so

$$9 - (3\sin x)^2 = 9\cos^2 x$$

So let $x = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{\sqrt{9-(3\sin\theta)^2}}{(3\sin\theta)^2}\, 3\cos\theta\, d\theta$$
$$= \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}\, 3\cos\theta\, d\theta$$
$$= \int \frac{\sqrt{9\cos^2\theta}}{9\sin^2\theta}\, 3\cos\theta\, d\theta$$

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{9\cos^2\theta}{9\sin^2\theta}\,d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int \cot^2\theta\,d\theta$$

$$= \int \csc^2\theta - 1\,d\theta$$

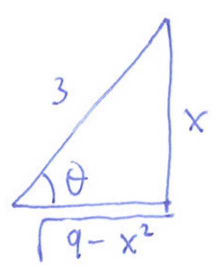
$$= \cot^2\theta - \theta + C$$

Now we want the answer in terms of x not θ. But we know

$$x = 3\sin\theta$$

$$\Rightarrow \sin\theta = \frac{x}{3}, \quad \theta = \arcsin\left(\frac{x}{3}\right)$$

Now using a triangle we can determin $\cot\theta$.



So $\cot\theta = \dfrac{\sqrt{9-x^2}}{x}$.

Thus $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \left(\frac{\sqrt{9-x^2}}{x}\right)^2 - \arcsin\left(\frac{x}{3}\right) + C$

$$= \frac{9-x^2}{x^2} - \arcsin\left(\frac{x}{3}\right) + C$$

eg Find $\int \frac{dx}{x^2\sqrt{x^2+3}}$

Let $x = \sqrt{3}\tan\theta$, $dx = \sqrt{3}\sec^2\theta \, d\theta$

$$\Rightarrow \int \frac{dx}{x^2\sqrt{x^2+3}} dx = \int \frac{\sqrt{3}\sec^2\theta \, d\theta}{(\sqrt{3}\tan\theta)^2\sqrt{(\sqrt{3}\tan\theta)^2+3}}$$

$$= \int \frac{\sqrt{3}\sec^2\theta \, d\theta}{3.\tan^2\theta\sqrt{3\tan^2\theta+3}}$$

$$= \int \frac{\sqrt{3}\sec^2\theta \, d\theta}{3\tan^2\theta\sqrt{3\sec^2\theta}}$$

$$= \frac{1}{3}\int \frac{\sec\theta}{\tan^2\theta} d\theta$$

$$= \frac{1}{3}\int \frac{\cos\theta}{\sin^2\theta} d\theta \quad , \text{ Since } \frac{\sec\theta}{\tan^2\theta} = \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}} = \frac{\cos\theta}{\sin^2\theta}$$

$$= \frac{1}{3}\int \frac{1}{u^2} du \quad , u = \sin\theta \, , du = \cos\theta \, d\theta$$

$$= -\frac{1}{3u} + C$$

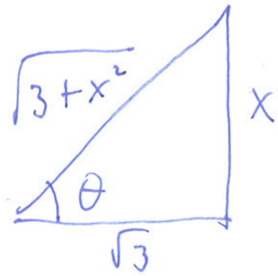
$$= -\frac{1}{3\sin\theta} + C$$

$$= -\frac{1}{3}\csc\theta + C$$

Again we want to use a triangle to determin $\csc\theta$.

$$x = \sqrt{3}\tan\theta \Rightarrow \tan\theta = \frac{x}{\sqrt{3}}$$

So



$$\Rightarrow \quad \csc\theta = \frac{\sqrt{3+x^2}}{x}$$

$$\Rightarrow \quad \int \frac{dx}{x^2\sqrt{x^2+3}} = \frac{-\sqrt{3+x^2}}{3x} + C$$

In general:

① If you see $a^2 - x^2$, then substitute $x = a\sin\theta$ since

$$a^2 - (a\sin\theta)^2 = a^2\cos^2\theta$$

② If you see $a^2 + x^2$, then substitute $x = a\tan\theta$ since

$$a^2 + (a\tan\theta)^2 = a^2\sec^2\theta$$

③ If you see $x^2 - a^2$, then substitute $x = a\sec\theta$ since

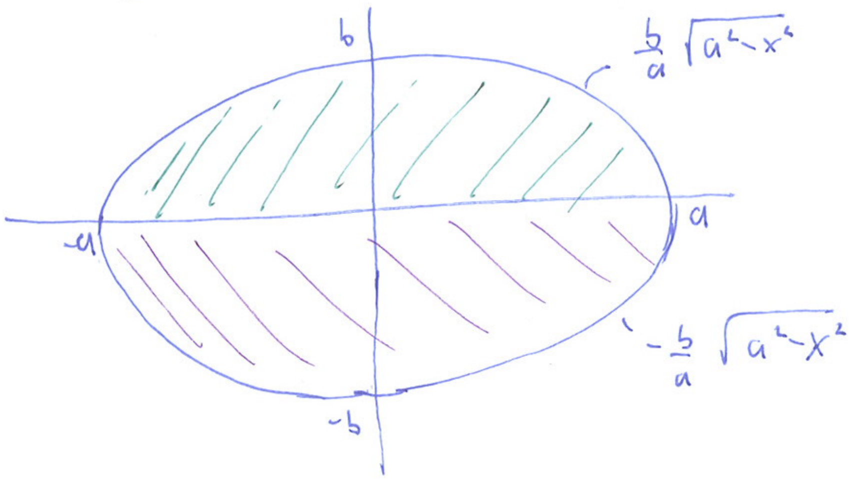
$$(a\sec\theta)^2 - a^2 = a^2\tan^2\theta$$

Now lets use trigsubs to find some definite integrals.

eg Find the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad , a, b > 0$$

We know the graph is:



Solving for y, we get

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$= \frac{b^2}{a^2}\left(a^2 - x^2\right)$$

$$\Rightarrow \quad y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

Let A = area of ellipse. Now by symetry we have

$$A = 2\int_{-a}^{a} \frac{b}{a}\sqrt{a^2-x^2}\, dx$$

Since the area under $\frac{b}{a}\sqrt{a^2-x^2}$ is equal (in size) of $-\frac{b}{a}\sqrt{a^2-x^2}$.

$$A = \frac{2b}{a} \int_{-a}^{a} \sqrt{a^2 - x^2}\, dx$$

Let $x = a \sin\theta$, $dx = a \cos\theta \, d\theta$.

Now before we make the substitution, note we need to take care of the limits of integration. We need to find what θ is when $x = \pm a$.

$$\begin{aligned} x = a \quad &\Rightarrow \quad a = a\sin\theta \\ &\Rightarrow \quad 1 = \sin\theta \\ &\Rightarrow \quad \theta = \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} x = -a \quad &\Rightarrow \quad -a = a\sin\theta \\ &\Rightarrow \quad -1 = \sin\theta \\ &\Rightarrow \quad \theta = -\frac{\pi}{2} \end{aligned}$$

Thus $A = \frac{2b}{a} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2 - (a\sin\theta)^2}\; a \cdot \cos\theta \, d\theta$

$$= \frac{2b}{a} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2 \cos^2\theta} \cdot a \cdot \cos\theta \, d\theta$$

$$= \frac{2b}{a} \cdot a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

$$= 2ab \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} (1 + \cos 2\theta) \, d\theta$$

$$= ab \left[\theta + \frac{\sin 2\theta}{2} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \pi ab$$

If $a = b = r$, then we get a circle of radius r, since:

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1 \Rightarrow x^2 + y^2 = r^2$$

Thus we have shown the area of a circle is πr^2!

One last example before we move on:

eg $$\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$$

As always we complete the square.

$$\begin{aligned} 3 - 2x - x^2 &= 3 - (x^2 + 2x) \\ &= 3 - (x^2 + 2x + 1 - 1) \\ &= 3 + 1 - (x^2 + 2x + 1) \\ &= 4 - (x+1)^2 \end{aligned}$$

$$\Rightarrow \int \frac{x}{\sqrt{3-2x-x^2}}\,dx = \int \frac{x}{\sqrt{4-(x+1)^2}}\,dx \quad , \; u = x+1, \; du = dx, \; x = u - 1$$

$$= \int \frac{u-1}{\sqrt{4-u^2}}\,du \quad , \; u = 2\sin\theta, \; du = \cos\theta\, d\theta$$

$$= \int \frac{(2\sin\theta - 1)}{\sqrt{4 - 4\sin^2\theta}} 2\cos\theta\, d\theta$$

$$= \int \frac{2\sin\theta - 1}{2\cos\theta} 2\cos\theta\, d\theta$$

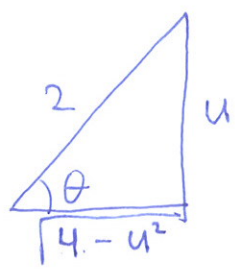
$$= \int 2\sin\theta - 1\, d\theta$$

$$\Rightarrow \int \frac{x}{\sqrt{3-2x-x^2}}\,dx = \int 2\sin\theta - 1 \, d\theta$$

$$= -2\cos\theta - \theta + C$$

Again lets find $\cos\theta$, θ:

$$u = 2\sin\theta \Rightarrow \sin\theta = \frac{u}{2}, \quad \theta = \arcsin\left(\frac{u}{2}\right)$$



So $\cos\theta = \frac{\sqrt{4-u^2}}{2}$

$$\Rightarrow \int \frac{x}{\sqrt{3-2x-x^2}}\,dx = \frac{-2\sqrt{4-u^2}}{2} - \arcsin\left(\frac{u}{2}\right) + C$$

$$= -\sqrt{4-(x+1)^2} - \arcsin\left(\frac{x+1}{2}\right) + C$$

$$= -\sqrt{3-2x-x^2} - \arcsin\left(\frac{x+1}{2}\right) + C$$